Weak Signal Detection by Small-Perturbation Control of Chaotic Orbits

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ABSTRACT

We have developed a method of detecting extremely weak signals using fundamental principles of chaotic dynamics. We calculate the time-dependent coupling of the disturbance signal to the stable and unstable manifolds of the system and derive a relationship between the disturbance and the error on the Poincaré surface of section, thus relating the disturbance signal to the perturbations needed to control a periodic orbit. We demonstrate the algorithm in a computer model of a chaotic system and discuss its implementation in high-frequency devices.

1. INTRODUCTION

Much interest has centered on the observation of chaotic behavior in dynamical systems. Since the demonstration by Ott, Grebogi, and Yorke that chaotic dynamics can be controlled using small perturbations[1], a large part of the research in chaos has been related to control, giving the promise of many useful technologies being built upon this foundation.

Three fundamental aspects of chaotic dynamics are as follows:[2] (1) a system behaving chaotically is exponentially sensitive to small changes, (2) within any chaotic attractor there exists a dense set of unstable periodic orbits, and (3) these periodic orbits can be controlled, or stabilized, using small perturbations. We make use of these three principles when developing this approach for detection.

2. MATHEMATICAL DESCRIPTION

A mathematical system which has become a paradigm for the study of chaotic processes is the Rössler system. Rössler developed this particular system of equations by studying the folding action which occurred in a mathematical system derived earlier by Lorenz[3]. It is described by a three dimensional system of nonlinear ordinary differential equations which are of the form,

$$\dot{x} = -(y+z)$$

$$\dot{y} = x + \alpha y$$

$$\dot{z} = \beta + (x-\mu)z.$$

The solutions of this system plotted parametrically in a cartesian coordinate system trace out a *state-space trajectory*. The structure in state-space that confines the trajectories is the state-space *attractor*. For particular values of the parameters α, β and μ, the system can have chaotic solutions which reside on an attractor of fractional dimension (a *strange attractor*). Figure 1 is a plot of one such attractor for parameters $\alpha = 0.2$, $\beta = 0.4$, and $\mu = 7.7$.

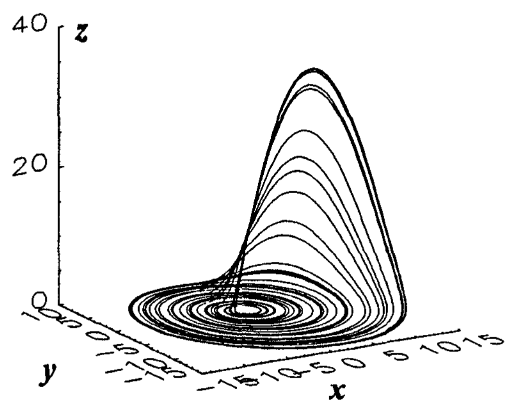

Figure 1. State-space attractor for the Rössler system for $\alpha = 0.2$, $\beta = 0.4$ and $\mu = 7.7$.

The trajectories are stretched upwards out of the *x-y* plane and then are folded back down on to the plane. This stretching and folding is typical of chaotic systems. This is descriptive of a *bounded instability* which is possible in nonlinear systems.

Controlling chaos, in the Ott, Grebogi, Yorke (OGY) sense, is to apply small perturbations to the system in order to stabilize one of the many unstable periodic orbits buried

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within the chaotic motion. It is advantageous to discretize the continuous time system by judicious placement of a two-dimensional surface which cuts through the trajectories. This surface is called the *Poincaré surface of section* (see fig. 2). This allows periodic trajectories, or orbits, to be described by points on the surface.

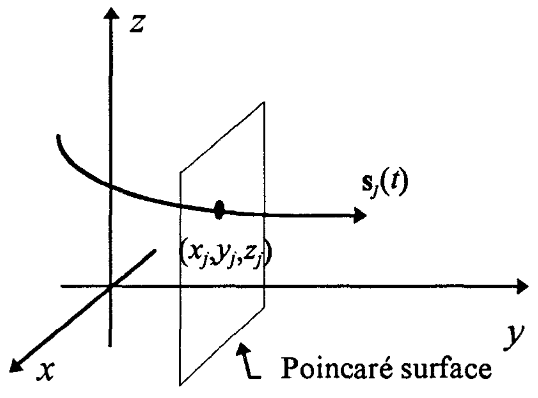

Figure 2. State-space trajectory intersecting the Poincaré surface of section.

We will designate $\left(x_j, y_j, z_j\right)$ as a period-j point on the Poincaré surface of section (PSS). A trajectory passing arbitrarily close to this point will follow a period-j orbit. The period of the oscillation, τ, is simply the time it takes the trajectory to make a complete orbit and return to the period-j point. In OGY control, we seek to cause the orbits to continuously pass through the periodic point(s) on the surface of section that we desire. The control perturbations are applied at the surface of section and are proportional to the error between the true trajectory crossing and the periodic point. If our system is described by $\dot{\mathbf{s}} = \mathbf{F}(\mathbf{s})$ then $\mathbf{p}_n \propto \varepsilon|_{\text{at PSS}} = \mathbf{s}|_{\text{at PSS}} - \mathbf{s}_j|_{\text{at PSS}}$.

If we assume that we will operate on or near periodic orbits we can analyze the stability of the system by solving the eigenvalue problem for the Jacobian matrix[2]. That is, $\mathbf{J}\mathbf{e} = \Lambda\mathbf{e}$, where

$$\mathbf{J} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & \alpha & 0 \\ z & 0 & x-\mu \end{bmatrix}.$$

The solution of $\det(\mathbf{J} - \mathbf{I}\Lambda) = 0$ gives a cubic equation of the form ,

$\Lambda^3 + a(x,z)\Lambda^2 + b(x,z)\Lambda + c(x,z) = 0$.

Notice that the Jacobian is dependent upon x and z, thus giving us time-varying stability solutions. We get eigenvalues $\Lambda_1, \Lambda_2, \Lambda_3$ and eigenvectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$. In general Λ_k can be complex. If $\mathrm{Re}(\Lambda_k) > 1$ anywhere then we have unstable motion along the trajectory. We can go further by casting the error in a basis derived from the eigenvectors. If $\mathbf{u}_k$ is a unit vector in the direction of the eigenvector $\mathbf{e}_k$, then we can find a component of the error in this eigen-direction by $\varepsilon_k = \varepsilon \cdot \mathbf{u}_k$. Now we have a component of the error in the direction of each eigenvector. We also want to quantify the expansion or compression of the error (stability) in terms of exponential coefficients. We can do so by letting $\lambda_k = \ln \Lambda_k$.

Our primary question is this: Can we determine a time-dependent disturbance signal from an accumulated error on the surface of section? If we have some time-dependent disturbance $\mathbf{d}(t)$ being coupled into the system then we also can find components of this disturbance in the eigen-directions as $d_k(t) = \mathbf{d}(t) \cdot \mathbf{u}_k(t)$. From an experimental standpoint, we would have to determine the strength and direction of the coupling through measurement. For the sake of a mathematical model we can simply say $\dot{\mathbf{s}} = \mathbf{F}(\mathbf{s}) + \mathbf{d}$. Since we are detecting extremely small signals, the addition of $\mathbf{d}$ does not significantly alter the system. For simplicity's sake we will consider the influence of $\mathbf{d}$ as coming from only one eigen-direction. We look at $d_k(t)$ for $t = 0$ to $t = \tau$. We can consider $d_k(t)$ as a superposition of Dirac delta functions in time (see fig. 3). In other words, $d_k(t) = \sum_{t'} D_{t'}\delta(t - t')$. If we derive an impulse response of the system to unit delta function, we can approach the analysis as being analogous to a Green's function problem. A unit delta function is expanded or compressed along the direction $\mathbf{u}_k$ exponentially by the factor λ_k. So then an impulse response in the direction $\mathbf{u}_k$ is just, $h_k(t) = e^{\lambda_k t}$. The component of the error with respect to a given periodic orbit due to a disturbance in an eigen-

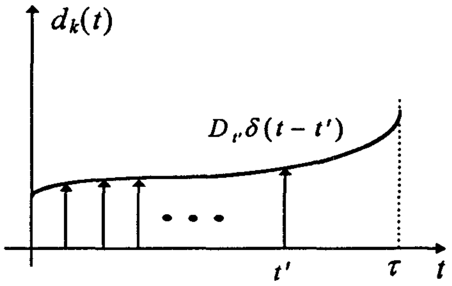

Figure 3. Disturbance signal $d_k(t)$ as a superposition of Dirac delta functions.

direction is determined by,

$\varepsilon_k(\tau) = d_k(t) * h_k(t) = \int_0^{\tau} d_k(t) e^{\lambda_k(\tau - t)} dt$. It is sufficient, although not complete, to consider only the direction of the error expansion (or the unstable direction, where $\mathrm{Re}(\lambda_k) > 0$). For application to devices, one would desire that the disturbance be coupled to the system primarily in the unstable direction.

We will demonstrate our idea using a linear disturbance described by, $\mathbf{d}(t) = (D_a t + D_b)\mathbf{u}_1$. We will assume that we have a constant, real expansion coefficient λ_1 in the $\mathbf{u}_1$ direction. So then

$$\varepsilon_1(\tau) = \int_0^{\tau} (D_a t + D_b) e^{\lambda_1(\tau - t)} dt,$$

$$= \gamma_a D_a + \gamma_b D_b, \text{ where}$$

$$\gamma_a = \frac{1}{\lambda_1^2}\left(e^{\lambda_1 \tau} - 1 - \lambda_1 \tau\right), \text{ and}$$

$$\gamma_b = \frac{1}{\lambda_1}\left(e^{\lambda_1 \tau} - 1\right).$$

Now $d_1(0) = D_b = \varepsilon_1(0)$. So then the disturbance, $d_1(t)$ is completely specified by the intercept $D_b = \varepsilon_1(0)$ and the slope,

$$D_a = \frac{\lambda_1^2}{\left(e^{\lambda_1 \tau} - 1 - \lambda_1 \tau\right)}\left[\varepsilon_1(\tau) - \frac{\left(e^{\lambda_1 \tau} - 1\right)}{\lambda_1}\varepsilon_1(0)\right]$$

As stated earlier, controlling chaotic trajectories through periodic orbits can be achieved by applying a small perturbation pulse at the surface of section. If we are considering only one direction, namely the unstable direction, then the perturbation applied in the unstable direction is proportional to the error in the unstable direction. For our example,

$$\mathbf{p}_{1_*} = \mathbf{p}_1(\tau) = p_1(\tau)\mathbf{u}_1 = \nu_1 \varepsilon_1(\tau)\mathbf{u}_1.$$

So then

$$D_a = \frac{\lambda_1^2}{\left(e^{\lambda_1 \tau} - 1 - \lambda_1 \tau\right)}\left[\frac{p_1(\tau)}{\nu_1} - \frac{\left(e^{\lambda_1 \tau} - 1\right)}{\lambda_1}\frac{p_1(0)}{\nu_1}\right]$$

and,

$D_b = \frac{1}{\nu_1} p_1(0)$. The disturbance signal is completely specified by the observed perturbation needed to maintain a periodic orbit. At each surface crossing the error is corrected by the perturbation and information about another τ-length section of the disturbance signal is given. After several crossings we can piece together the complete disturbance signal.

We have developed the basis for completely resolving a time-dependent disturbance signal that is *slowly varying* compared to the period of the chaotic oscillator. Signals not slowly varying can be resolved through methods such as: strict de-convolution, higher-order disturbance signal approximation functions, gating at the surface of section which would lead to near-Nyquist sampling limitations. We are exploring these issues at this time.

From an experimental and/or design standpoint, the time-dependent expansion-compression coefficients and their corresponding directions can be determined through measurement. Analog techniques have been developed which allow control and stabilization of periodic orbits in high-frequency chaotic devices.[4] In the microwave regime, similar techniques can be used for the design of microwave frequency detection (sensor) devices.

3. MODEL RESULTS

We developed a computer model using a chaotic system derived from an electrical oscillator. The double-scroll oscillator is a simple negative differential resistance oscillator circuit which can be described by the following system of equations:

$$C_1\dot{v}_{C_1} = (v_{C_2} - v_{C_1})G - g(v_{C_1})$$
$$C_2\dot{v}_{C_2} = (v_{C_1} - v_{C_2})G + i_L$$
$$L\dot{i}_L = -v_{C_2}$$

where $g(v)$ is a piece-wise negative differential resistance element. The circuit is capable of Rössler-type behavior for certain values of C_1, C_2, L and G. We applied control perturbations in the unstable direction to control a period-one orbit (see fig. 4).

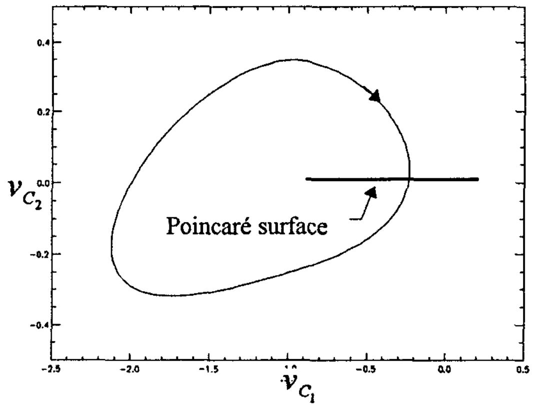

Figure 4. Period-one orbit controlled using perturbations applied at $v_{C_2} = 0$.

We also implemented a tracking algorithm which sequentially corrected the periodic-point on the surface of section. This served to significantly reduce the magnitude of the control perturbations. At the 50th surface crossing we turned off the tracking and turned on a sinusoidal disturbance signal $d(t) = A\sin\left(\frac{2\pi}{\tau_d}t\right)$, where $A = 10^{-11}$ and $\tau_d = 100\tau$. The period τ_d was sufficient to give an approximately linear relationship to $d(t)$ from surface crossing to surface crossing. Figure 5 shows the results of the simulation. It shows that the influence of the disturbance signal is impressed upon the control perturbations. The expression derived earlier gives a scaling function which will return the actual value of the disturbance.

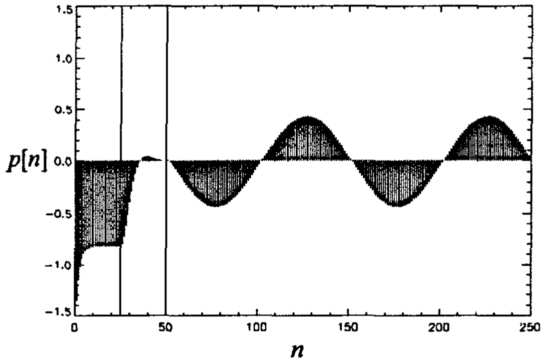

Figure 5. Control perturbations ($\times 10^{-11}$) versus surface of section crossing. Disturbance turned on at $n = 50$.

4. CONCLUSION

We have described the basic formalism for detection of weak signals using the controlled orbits of a chaotic system. We make use of the inherent exponential sensitivity of chaotic dynamics to small changes, the density of unstable periodic orbits within the chaotic behavior, and the techniques developed to control chaotic trajectories through these periodic orbits. We demonstrated the detection using a period-one orbit. Making use of higher period orbits would bring greater sensitivity and flexibility to a proposed detection device. Chaotic behavior has been observed and quantified experimentally in different microwave systems.[5,6] The techniques outlined here are general in nature and can be implemented in microwave systems using analog control techniques. We must also explore the fundamental limitations upon disturbance signal variation with respect to the period of the controlled chaotic orbit. Therein lies elements of sampling and filter theory.

5. REFERENCES

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